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ITERATED CIRCLE BUNDLES AND INFRANILMANIFOLDS

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Abstract
We give short proofs of the following two facts: Iterated principal circle bundles are precisely the nilmanifolds. Every iterated circle bundle is almost flat, and hence diffeomorphic to an infranilmanifold.

An *infranilmanifold* is a closed manifold diffeomorphic to the quotient space N/Γ of a simply-connected nilpotent Lie group N by a discrete torsion-free subgroup Γ of the semidirect product $N \rtimes C$ where C is a maximal compact subgroup of $\mathrm{Aut}(N)$. If Γ lies in the N factor, the infranilmanifold is called a *nilmanifold.*

An *iterated circle bundle* is defined inductively as the total space of a circle bundle whose base is an iterated circle bundle of one dimension lower, and the base at the first step is a point. If at each step the circle bundle is principal, the result is an *iterated principal circle bundle.*

This note was prompted by a question of Xiaochun Rong who asked me to justify the following fact mentioned in [1]:

Theorem 1. *A manifold is an iterated principal circle bundle if and only if it is a nilmanifold.*

The proof of Theorem 1 combines some bundle-theoretic considerations with classical results of Mal'cev [8]. The "if" direction was surely known since [8] but [3, Proposition 3.1] seems to be the earliest reference. The statement of Theorem 1 is mentioned without proof in [14, p.98] and [4, p.122].

Summary of previous work:

(1) Every iterated principal circle bundle has torsion-free nilpotent fundamental group because the homotopy exact sequence converts a principal circle bundle into a central extension with infinite cyclic kernel.

(2) Theorem 1.2 of [9] implies that every iterated principal circle bundle is diffeomorphic to an infranilmanifold; this was explained to me by Xiaochun Rong. Thus [9] gives another (less elementary) proof of the "only if" direction in Theorem 1 because every iterated principal circle bundle is homotopy equivalent to a nilmanifold, and the diffeomorphism type of an infranilmanifold is determined by its homotopy type [7].

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(3) According to [10] a manifold is a principal torus bundle over a torus if and only if it is a nilmanifold modelled on a two-step nilpotent Lie group.

(4) Every 3-dimensional infranilmanifold has a unique Seifert fiber space structure, see [13, Theorem 3.8], hence it is an iterated circle bundle if and only if the base orbifold (of the Seifert fibering) is non-singular, i.e., the 2-torus or the Klein bottle. Thus some 3-dimensional infranilmanifolds are not iterated circle bundles.

(5) In [6] it is proven that every iterated circle bundle is homeomorphic to an infranilmanifold. Their argument splits in two parts: finding a homotopy equivalence and upgrading it to a homeomorphism. The latter uses topological surgery, which does not extend to the smooth setting.

(6) A natural way to establish the smooth version of the above-mentioned result in [6] is to show that every iterated circle bundle is almost flat, and then apply the celebrated work of Gromov-Ruh [5, 12] that infranilmanifolds are precisely the almost flat manifolds. Recall that a closed manifold is *almost flat* if it admits a sequence of Riemannian metrics of uniformly bounded diameters and sectional curvatures approaching zero. To this end we prove:

Theorem 2. *Any iterated circle bundle is almost flat, and therefore diffeomorphic to an infranilmanifold.*

Proof of Theorem 1. We use [11, Chapter II] as a reference for Mal'cev's work. If N/Γ is a nilmanifold, then Γ is finitely generated, torsion-free, and nilpotent, and conversely, any such group is the fundamental group of a nilmanifold, see [11, Theorem 2.18]. Every automorphism of Γ extends uniquely to an automorphism of N, see [11, Theorem 2.11]. Applying this to conjugation by an element of the center of Γ we get the inclusion of centers $Z(\Gamma) \subset Z(N)$. Nilpotency of Γ ensures that $Z(\Gamma)$ is nontrivial, and therefore, there is a one-parameter subgroup $R \leq Z(N)$ such that $R \cap Z(\Gamma)$ is nontrivial, and hence infinite cyclic. Clearly $R \cap \Gamma = R \cap Z(\Gamma)$. The left R-action on N descends to a free $R/(R \cap \Gamma)$-action on N/Γ, which makes N/Γ into a principal circle bundle whose base B_Γ is a nilmanifold, namely, the quotient of N/R by $\Gamma/(R \cap \Gamma)$. This proves the "if" direction.

Conversely, let $p\colon E \to B$ be a principal circle bundle over a nilmanifold B. Its homotopy exact sequence is a central extension, so $\pi_1(E)$ is finitely generated torsion-free nilpotent. Consider a nilmanifold N/Γ with $\Gamma \cong \pi_1(E)$, and let $z \in Z(\Gamma)$ be the element corresponding to the circle fiber of p through the basepoint. Let $R \leq N$ be the one-parameter subgroup that contains z. As above $R \subset Z(N)$ and N/Γ is the total space of a principal circle bundle $p_\Gamma\colon N/\Gamma \to B_\Gamma$ whose base B_Γ is a nilmanifold and the fibers are the $R/(R \cap \Gamma)$-orbits. The cyclic group $R \cap \Gamma$ is generated by z because its generator projects to a finite order element in the torsion-free group $\Gamma/\langle z\rangle \cong \pi_1(B)$. Thus the isomorphism $\pi_1(E) \cong \pi_1(N/\Gamma)$ descends to an isomorphism $\pi_1(B) \to \pi_1(B_\Gamma)$. Since all these manifolds are aspherical, the fundamental group isomorphisms are induced by homotopy equivalences, and we get a homotopy-commutative square

$$\begin{array}{ccc} E & \xrightarrow{\varepsilon} & N/\Gamma \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_\Gamma} \\ B & \xrightarrow[\beta]{} & B_\Gamma \end{array}$$

where ε and β are homotopy equivalences. We can assume that β is a diffeomorphism because by [11, Theorem 2.11] any homotopy equivalence of nilmanifolds is homotopic to a diffeomorphism. The Gysin sequence implies that the Euler class of a circle bundle generates the kernel of the homomorphism induced on the second cohomology by the bundle projection. The map of the Gysin sequences of p and p_Γ induced by the commutative square shows that β preserves their Euler classes up to sign, and after changing the orientation if necessary we can assume that the Euler classes are preserved by β. The isomorphism type of a principal circle bundle is determined by its Euler class. Since p and the pullback of p_Γ via β have the same Euler class, they are isomorphic, which gives a desired diffeomorphism of E and N/Γ and completes the proof of the "only if" direction. □

Proof of Theorem 2. In view of [5, 12] it is enough to prove inductively that the total space of any circle bundle over an almost flat manifold is almost flat. This comes via the following standard argument. Let $p\colon E \to B$ be a smooth circle bundle over a closed manifold B. For any Riemannian metric $\check{g}$ on B there is a metric g on E such that p is a Riemannian submersion with totally geodesic fibers which are isometric to the unit circle, see [2, 9.59]. As in [2, 9.67] let g^t be the metric on E obtained by rescaling g by a positive constant t along the fibers of p, i.e., g^t and g have the same vertical and horizontal distributions $\mathcal{V}$, $\mathcal{H}$, and $g^t|_{\mathcal{V}} = tg|_{\mathcal{V}}$ and $g^t|_{\mathcal{H}} = g|_{\mathcal{H}}$. The fibers of p are g^t-totally geodesic [2, 9.68] so the T tensor vanishes. The diameters of g^t, $\check{g}$ satisfy $\mathrm{diam}(g^t) \le \mathrm{diam}(\check{g}) + O(\sqrt{t})$. The following lemma finishes the proof of almost flatness of E. □

Lemma 3. *The sectional curvatures K^t, $\check{K}$ of g^t, $\check{g}$ satisfy $|K^t| \le |\check{K}| + O(\sqrt{t})$.*

Proof. Fix any 2-plane σ tangent to E. Since $\mathcal{H}$ has codimension one, σ contains a g^t-unit horizontal vector X. Let C be a g^t-unit vector in σ that is g^t-orthogonal to X. Write $C = U + Y$ where $U \in \mathcal{V}$, $Y \in \mathcal{H}$. The sectional curvature of σ with respect to g^t is given by

$$K^t_\sigma = \langle R^t(C,X)C,X\rangle^t = \langle R^t(Y,X)Y,X\rangle^t + 2\langle R^t(Y,X)U,X\rangle^t + \langle R^t(U,X)U,X\rangle^t$$

where $\langle C,D\rangle^t := g^t(C,D)$ and R^t is the curvature tensor of g^t.

Lemma 9.69 of [2] relates the A tensors A^t, A of g^t, g as follows: $A^t_Y X = A_Y X$ and $A^t_X U = t A_X U$. Recall that $A_Y X$ is vertical and $A_X U$ is horizontal. The formulas in [2, 9.28, 9.69] give

$$\check{g}(\check{R}(\check{Y},\check{X})\check{Y},\check{X}) - \langle R^t(Y,X)Y,X\rangle^t = 3\langle A^t_Y X, A^t_Y X\rangle^t = 3t\, g(A_Y X, A_Y X)$$

$$\langle R^t(Y,X)U,X\rangle^t = -[\langle (D_X A)_Y X, U\rangle]^t = -t\, g((D_X A)_Y X, U)$$

$$\langle R^t(U,X)U,X\rangle^t = \langle A^t_X U, A^t_X U\rangle^t + [\langle (D_U A)_X X, U\rangle]^t = t^2 g(A_X U, A_X U)$$

where $[\langle (D_U A)_X X, U\rangle]^t = 0$ by the last formula in [2, 9.32].

Since $g(X,X) = 1 = g^t(C,C) = g(Y,Y) + tg(U,U)$, the vectors X, Y, $\sqrt{t}\,U$ lie in the g-unit disk bundle of TE, which is compact, so the functions $g(A_Y X, A_Y X)$, $\sqrt{t}\,g((D_X A)_Y X, U)$, $t\,g(A_X U, A_X U)$ are bounded.

Therefore, if $Y \neq 0$ and $\check{\sigma}$ is the projection of σ in TB, then

$$K^t_\sigma = \check{g}(\check{R}(\check{Y}, \check{X})\check{Y}, \check{X}) + O(\sqrt{t}) = \sqrt{\check{g}(\check{Y}, \check{Y})}\,K_{\check{\sigma}} + O(\sqrt{t})$$

and if $Y = 0$, then $K^t_\sigma = t^2 g(A_X U, A_X U) = O(t)$. Thus $|K^t_\sigma| \leq |K_{\check{\sigma}}| + O(\sqrt{t})$. □

References

[1] I. Belegradek and G. Wei: *Metrics of positive Ricci curvature on vector bundles over nilmanifolds*. Geom. Funct. Anal. **12** (2002), 56–72.

[2] A.L. Besse: Einstein manifolds, Classics in Mathematics. Springer-Verlag, Berlin, 2008, Reprint of the 1987 edition.

[3] E. Fadell and S. Husseini: *On a theorem of Anosov on Nielsen numbers for nilmanifolds*; in Nonlinear functional analysis and its applications (Maratea, 1985), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **173**, Reidel, Dordrecht, 47–53, 1986.

[4] Y. Félix, J. Oprea, and D. Tanré: Algebraic models in geometry, volume 17 of Oxford Graduate Texts in Mathematics **17**, Oxford University Press, Oxford, 2008.

[5] M. Gromov: Almost flat manifolds. J. Differential Geom. **13** (1978), 231–241.

[6] J.B. Lee and M. Masuda: *Topology of iterated S^1-bundles*, Osaka J. Math. **50** (2013), 847–869.

[7] K.B. Lee and F. Raymond: *Geometric realization of group extensions by the Seifert construction.*; in Contributions to group theory, Contemp. Math, Amer. Math. Soc., Providence, RI, 353–411, 1984.

[8] A.I. Mal'cev: *On a class of homogeneous spaces*, Izvestiya Akad. Nauk. SSSR. Ser. Mat., **13** (1949), 9–32.

[9] M. Nakayama: *On the S^1-fibred nilBott tower*, Osaka J. Math. **51** (2014), 67–87, 2014.

[10] R.S. Palais and T.E. Stewart: *Torus bundles over a torus*, Proc. Amer. Math. Soc, **12** (1961), 26–29.

[11] M.S. Raghunathan: Discrete subgroups of Lie groups, Springer-Verlag, New York-Heidelberg, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **68**, 1972.

[12] E.A. Ruh: *Almost flat manifolds*, J. Differential Geom, **17** (1982), 1–14.

[13] P. Scott: *The geometries of 3-manifolds*, Bull. London Math. Soc, **15** (1983), 401–487.

[14] S. Weinberger: The topological classification of stratified spaces. Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1994.

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